Remarks on the spherical waves of the Dirac field on de Sitter spacetime

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Abstract

The Shishkin's solutions of the Dirac equation in spherical moving frames of the de Sitter spacetime are investigated pointing out the set of commuting operators whose eigenvalues determine the integration constants. It is shown that these depend on the usual angular quantum numbers and, in addition, on the value of the scalar momentum. With these elements a new result is obtained finding the system of solutions normalized (in generalized sense) in the scale of scalar momentum.

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The Dirac equation on de Sitter spacetime M (whose cosmological constant Λ_c gives the radius $R = 1/\omega = \sqrt{3/\Lambda_c}$,) has been studied in moving or static local charts (i.e., natural frames) suitable for separation of variables leading to significant analytical solutions [1]-[5].

The first spinor solutions on M were obtained in a *static* central chart with spherical coordinates, $\{t_s, r_s, \theta, \phi\}$, using the diagonal tetrad gauge [1]. Another gauge of Cartesian type leads to a system of particular solutions of the Dirac equation written in terms of the well-known spherical spinors of special relativity [3]. All these particular solutions are eigenspinors of the Hamiltonian operator $H = i\partial_{t_s}$ which has a continuous energy spectrum. The form of these solutions is too complicated such that the normalization in the energy scale can not be done.

Of a particular interest is the *moving* frame with the proper time and spherical space coordinates, $\{t, r, \theta, \phi\}$, associated to the Cartesian chart $\{t, \vec{x}\}$ with the line element [6]

$$ds^2 = dt^2 - e^{2\omega t} d\vec{x}^2 \,. \tag{1}$$

A set of particular spherically symmetric solutions of the Dirac field in the chart $\{t, r, \theta, \phi\}$ was found by Shishkin [2] using a Cartesian tetrad gauge and a suitable method of separating variables [7]. We note that in the moving charts the operator $i\partial_t$ is no longer a Killing vector field and, therefore, the separation of variables leads to solutions which are no energy eigenspinors, their integration constants depending on other physical quantities. However, to our knowledge, a complete system of normalized spherical wave solutions of the Dirac equation was not constructed so far.

The plane wave solutions in the diagonal gauge of the chart $\{t, \vec{x}\}$ were obtained in [8, 9] but these solutions are normalized only in the asymptotic approximation. The first complete system of normalized plane wave solutions in this chart was derived using a suitable complete set of commuting operators representing conserved observables associated with the specific isometries of the de Sitter manifolds [4]. In this manner all the integration constants were determined as eigenvalues of the operators of this set, finding the form of the particular solutions that can be normalized in generalized sense in the momentum scale.

In this letter we would like to apply the same method for the spherical waves in the chart $\{t, r, \theta, \phi\}$ trying to construct normalized solutions as linear combinations of the Shishkin's ones. Our approach is based on the theory

of external symmetry [10] which explains the relations among the geometric symmetries and the operators commuting with the Dirac one, constructed with the help of the Killing vectors some time ago [11]. In fact, these operators are nothing other than the generators of the spinor representation of the universal covering group of the isometry group [10] and, therefore, they constitute the main physical observables among which we can choose different sets of commuting operators defining quantum modes. This method is efficient especially in the case of the de Sitter spacetime where the high symmetry given by the $SO(4,1)$ isometry group [12, 13] offers one the opportunity of a rich algebra of conserved operators able to receive a physical meaning.

Given an arbitrary chart $\{x\}$ of M with coordinates x^μ $(\mu,\nu,...=0,1,2,3)$, we have to choose the tetrad fields, $e_{\hat\mu}(x)$ and $\hat e^{\hat\mu}(x)$ which define the local (unholonomic) frames and the corresponding coframes. These are labeled by the local indices, $\hat\mu,\hat\nu,...=0,1,2,3$, and have the orthonormalization properties [4] with respect to the flat metric $\eta=$diag$(1,-1,-1,-1)$. The metric tensor of M, $g_{\mu\nu}=\eta_{\hat\alpha\hat\beta}\hat e^{\hat\alpha}_\mu \hat e^{\hat\beta}_\nu$, raises or lowers the Greek indices while for the local indices (with hat) we have to use the flat metric. In what follows we consider the Cartesian moving charts $\{t,\vec x\}$, and $\{t_c,\vec x\}$ as well as the corresponding spherical ones, $\{t,r,\theta,\phi\}$ and $\{t_c,r,\theta,\phi\}$. Here t_c is the conformal time defined by

$$\omega t_c = -\, e^{-\omega t}\,, \tag{2}$$

which gives the simpler line element [6],

$$ds^2=\frac{1}{\omega^2 {t_c}^2}\left(d{t_c}^2-d\vec x^2\right)\,. \tag{3}$$

We note that the coordinates of the moving frames are related to those of the static ones through $e^{-\omega t}=e^{-\omega t_s}\cosh\omega r_s$ and $r=e^{-\omega t_s}\sinh\omega r_s$ while the angular variables remain the same.

The theory of the Dirac field ψ minimally coupled with gravitation can be written simply in the Cartesian gauge where the non-vanishing tetrad components are [2]

$$e^0_0=-\omega t_c\,,\quad e^i_j=-\delta^i_j\,\omega t_c\,,\quad \hat e^0_0=-\frac{1}{\omega t_c}\,,\quad \hat e^i_j=-\delta^i_j\,\frac{1}{\omega t_c}\,. \tag{4}$$

In this gauge the Dirac operator reads [4]

$$E_D \;=\; -i\omega t_c\left(\gamma^0\partial_{t_c}+\gamma^i\partial_i\right)+\frac{3i\omega}{2}\gamma^0$$

$$= \; i\gamma^0\partial_t + ie^{-\omega t}\gamma^i\partial_i + \frac{3i\omega}{2}\gamma^0\,, \tag{5}$$

where the γ matrices satisfy $\{\gamma^{\hat\alpha},\gamma^{\hat\beta}\} = 2\eta^{\hat\alpha\hat\beta}$ and $S^{\hat\mu\hat\nu} = \frac{i}{4}[\gamma^{\hat\mu},\gamma^{\hat\nu}]$. The conserved operators commuting with E_D are the generators of the external symmetry group $S(M)$ which is just the universal covering group of the isometry group $I(M) = SO(4,1)$ [10]. In Ref. [4] we pointed out that the Hamiltonian operator H, the components of the *momentum*, P^i, and those of the total angular momentum, $J^i = \varepsilon_{ijk}J_{jk}/2$, are the following basis-generators of $S(M)$

$$P^i = -i\partial_i \tag{6}$$
$$H = -i\omega(t_c\partial_{t_c} + x^i\partial_i) \tag{7}$$
$$J_{ij} = -i(x^i\partial_j - x^j\partial_i) + S_{ij}\,, \tag{8}$$

remaining with three more basis-generators which do not have an immediate physical significance [4]. In the chart, $\{t,\vec x\}$, the operators $\vec P$ and $\vec J = \vec L + \vec S$ (with $\vec L = \vec x \times \vec P$) keep their forms while the Hamiltonian operator becomes $H = i\partial_t + \omega\,\vec x\cdot\vec P$. We observe that the presence of the external gravitational field ($\omega \neq 0$) leads to the commutation rules

$$[H,\, P^i] = i\omega P^i\,, \tag{9}$$

which prevent one to diagonalize simultaneously the operators H and P^i. For this reason, the plane wave solutions of [4] were derived as common eigenspinors of the complete set of commuting operators $\{E_D, P^i, W = \vec P\cdot\vec S\}$.

In these circumstances it is natural to consider the spherical modes defined by the common eigenspinors of the complete set $\{E_D, \vec P^2, \vec J^2, K, J_3\}$ where $\vec P^2$ plays the role of H in static frames [3]. We remind the reader that the operator $K = \gamma^0(2\vec L\cdot\vec S + 1)$ concentrates the action of all the angular operators. Our purpose is to write down the particular solutions of the Dirac equation $E_D\psi = m\psi$, of mass m, in such a way that $\psi = \psi_{p,\kappa_j,m_j}$ be a common eigenspinor of the above set of commuting operators, corresponding to the eigenvalues $\{m, p^2, j(j+1), -\kappa_j, m_j\}$ where $\kappa_j = \pm(j+\frac{1}{2})$ [14] while p is the value of the *scalar momentum*. In addition, we require these solutions to be normalized with respect to the time-independent relativistic scalar product defined as [4]

$$\langle\psi,\psi'\rangle = \int d^3x\, e^{3\omega t}\,\overline{\psi}(t,\vec x)\gamma^0\psi'(t,\vec x)\,, \tag{10}$$

in the chart $\{t, \vec{x}\}$.

For solving the above eigenvalue problems it is convenient to start with the chart $\{x\} = \{t_c, r, \theta, \phi\}$ looking for particular solutions of the form

$$\psi_{p,\kappa_j,m_j}(x) = \frac{(-\omega t_c)^{\frac{3}{2}}}{r}\left[f^+_{p,\kappa_j}(t_c,r)\Phi^+_{m_j,\kappa_j}(\theta,\phi) + f^-_{p,\kappa_j}(t_c,r)\Phi^-_{m_j,\kappa_j}(\theta,\phi)\right] \tag{11}$$

where $\Phi^{\pm}_{m_j,\kappa_j}$ are the usual normalized spherical spinors of special relativity that solve the eigenvalue problems of the operators $\vec{J}^2$, K and J_3 [14]. Then, denoting by $k = m/\omega$, after a little calculation, we arrive ar the pair of equations

$$\left(\pm i\partial_{t_c} + \frac{k}{t_c}\right) f^{\pm}_{p,\kappa_j}(t_c,r) = \left(-\partial_r \pm \frac{\kappa_j}{r}\right) f^{\mp}_{p,\kappa_j}(t_c,r)\,, \tag{12}$$

resulted from the Dirac one. In addition, the eigenvalue problem of $\vec{P}^2$ leads to the supplemental radial equations

$$\left[-\partial_r^2 + \frac{\kappa_j(\kappa_j \pm 1)}{r^2}\right] f^{\pm}_{p,\kappa_j}(t_c,r) = p^2 f^{\pm}_{p,\kappa_j}(t_c,r)\,, \tag{13}$$

since the spinors $\Phi^{\pm}_{m_j,\kappa_j}$ are eigenfunctions of $\vec{L}^2$ corresponding to the eigenvalues $\kappa_j(\kappa_j \pm 1)$. Eqs. (12) and (13) can be solved separating the variables,

$$f^{\pm}_{p,\kappa_j}(t_c,r) = \tau^{\pm}_p(t_c)\rho^{\pm}_{p,\kappa_j}(r)\,, \tag{14}$$

and finding that the new functions must satisfy

$$\left(\pm i\partial_{t_c} + \frac{k}{t_c}\right)\tau^{\pm}_p(t_c) = \pm p\,\tau^{\mp}_p(t_c)\,, \tag{15}$$

$$\left(\pm\partial_r + \frac{\kappa_j}{r}\right)\rho^{\pm}_{p,\kappa_j}(r) = p\,\rho^{\mp}_{p,\kappa_j}(r)\,. \tag{16}$$

These equations have to be solved in terms of Bessel functions [15] as in [2]. The advantage of our method is to point out that there is only one additional integration constant, p, which is a continuous parameter with a precise physical meaning (i.e., the scalar momentum).

However, our main problem is to find the normalized solutions with respect to the scalar product (10). We specify that these solutions are not

square integrable since the spectrum of $\vec{P}^2$ is continuous. Therefore, we must look for a system of spinors ψ_{p,κ_j,m_j} normalized in the generalized sense in the scale of the scalar momentum p. First, we choose the radial functions as in [2],

$$\rho^{\pm}_{p,\kappa_j}(r) = \sqrt{pr}\, J_{|\kappa_j \pm \frac{1}{2}|}(pr)\,. \tag{17}$$

Furthermore, we observe that the functions $\tau^{\pm}$ must produce a similar time modulation as in the case of the normalized plan wave solutions of [4]. Consequently, we denote $\nu_{\pm} = \frac{1}{2} \pm ik$ and write the functions $\tau^{\pm}$ in terms of Hankel functions [15] as

$$\tau^{\pm}_p(t_c) = N\sqrt{-pt_c}\, e^{\pm\pi k/2} H^{(1)}_{\nu_{\mp}}(-pt_c) \tag{18}$$

where N is a normalization factor. Finally, summarizing these results in the chart $\{x\} = \{t, r, \theta, \phi\}$ and matching the factor N we find the definitive form of the normalized spinors

$$\begin{aligned}\psi_{p,\kappa_j,m_j}(x) = \frac{p}{2}\sqrt{\frac{\pi}{\omega r}}\, e^{-2\omega t}\, \Big[& e^{\pi k/2} H^{(1)}_{\nu_-}(\tfrac{p}{\omega}e^{-\omega t}) J_{|\kappa_j + \frac{1}{2}|}(pr) \Phi^{+}_{m_j,\kappa_j}(\theta,\phi) \\ & + e^{-\pi k/2} H^{(1)}_{\nu_+}(\tfrac{p}{\omega}e^{-\omega t}) J_{|\kappa_j - \frac{1}{2}|}(pr) \Phi^{-}_{m_j,\kappa_j}(\theta,\phi) \Big]\end{aligned} \tag{19}$$

that satisfy the Dirac equation and are common eigenspinors of the operators $\vec{P}^2, \vec{J}^2, K$ and J_3. Taking into account that [16]

$$\int_0^\infty \rho_{p,\kappa_j}(pr)\rho_{p',\kappa_j}(p'r)dr = \delta(p - p')\,, \tag{20}$$

and using the properties of Hankel functions mentioned in [4] we obtain the orthonormalization rule

$$\left\langle \psi_{p,\kappa_j,m_j}, \psi_{p',\kappa'_{j'},m'_j} \right\rangle = \delta(p - p')\delta_{j,j'}\delta_{\kappa_j,\kappa'_j}\delta_{m_j,m'_j}\,. \tag{21}$$

According to the conventions of [4] we can say that the particular solutions (19) are of positive frequencies and, therefore, these describe the quantum modes of the Dirac particles. The spinors of negative frequencies, corresponding to antiparticles, will be obtained using the charge conjugation as in [4]. In this way one may obtain a complete system of orthonormalized spinors which could be the starting point to the canonical quantization of the Dirac field in spherical moving frames.

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